Floyd-Warshall Algorithm

Lecture 09(a)

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1. All Pair Shortest Path Distance
   - The Floyd-Warshall Algorithm

2. Transitive Closure
   - The adapted Floyd-Warshall Algorithm

Outline

Introduction

Problem

Given a Weighted Directed Graph $G = (V, E)$ with a weight function $w : E \to \Re$ where $\Re$ is the set of real numbers, find the length of the **shortest path** between all pairs of vertices in G.

Input

Given: $G = (V, E)$ and $w : E \to \Re$ such that $|V| = n$.
Format: G as adjacency matrix of weights W such that

$$W[i][j] = w_{ij} \begin{cases} 0 & \text{if } (i = j) \\ \text{weight on edge } (i, j) & \text{if } i \neq j \text{ and } (i, j) \in E \\ \infty & \text{if } i \neq j \text{ and } (i, j) \notin E \end{cases}$$

Output

$n \times n$ matrix D such that $D[i][j] = d_{ij}$ where d_{ij} is the length of the shortest path (or shortest distance) between the vertices i and j.

Assumption: no negative cycle in G.

Notations

Definition: Intermediate Vertices

For a path $P = \langle v_1, v_2, \cdots, v_l \rangle$, an **intermediate vertex** is any vertex of P other than v_1 or v_l.

d_{ij}^k

- Let d_{ij}^k = shortest distance (path-weight) of any path $i \rightsquigarrow j$ with all intermediate vertices in $\{1, 2, \cdots, k\}$.
- $d_{ij}^0 = w_{ij}$ as no intermediate vertices.
- d_{ij}^n = shortest distance (path-weight) between i and j.

D^k

- $D^k = [d_{ij}^k]$
- $D^0 = W$
- $D^n = D$

Recurrence Relation

Consider a shortest path $i \overset{P}{\rightsquigarrow} j$ such that all intermediate vertices are in $\{1, 2, \cdots, k\}$ for $(k \geq 1)$
2 possibilities:

- k is not an intermediate vertex
  $\Rightarrow P$ has shortest distance $= d_{ij}^{k-1}$.
- k is an intermediate vertex
  $\Rightarrow i \overset{p_1}{\rightsquigarrow} k \overset{p_2}{\rightsquigarrow} j$ such that all intermediate vertices of p_1 and p_2 are in $\{1, 2, \cdots, k-1\}$
  $\Rightarrow P$ has shortest distance $= d_{ik}^{k-1} + d_{kj}^{k-1}$.

For $k \geq 1$
$d_{ij}^{k} = \min(d_{ij}^{k-1}, d_{ik}^{k-1} + d_{kj}^{k-1})$

Recurrence Relation

Consider a shortest path $i \overset{P}{\rightsquigarrow} j$ such that all intermediate vertices are in $\{1, 2, \cdots, k\}$ for $(k \geq 1)$
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  $\Rightarrow P$ has shortest distance $= d_{ik}^{k-1} + d_{kj}^{k-1}$.

For $k \geq 1$
$d_{ij}^{k} = \min(d_{ij}^{k-1}, d_{ik}^{k-1} + d_{kj}^{k-1})$

Pseudocode

Algorithm 1 Floyd-Warshall Algo

procedure ALLPAIRSHORTESTDISTANCE(W, n)
  $D^0 \leftarrow W$ *** Initialization ***
  for $k \leftarrow 1$ to n **do**
    for $i \leftarrow 1$ to n **do**
      for $j \leftarrow 1$ to n **do**
        $d_{ij}^k \leftarrow \min(d_{ij}^{k-1}, d_{ik}^{k-1} + d_{kj}^{k-1})$
      end for
    end for
  end for
  Return D^n
end procedure

Pseudocode

Algorithm 2 Floyd-Warshall Algo

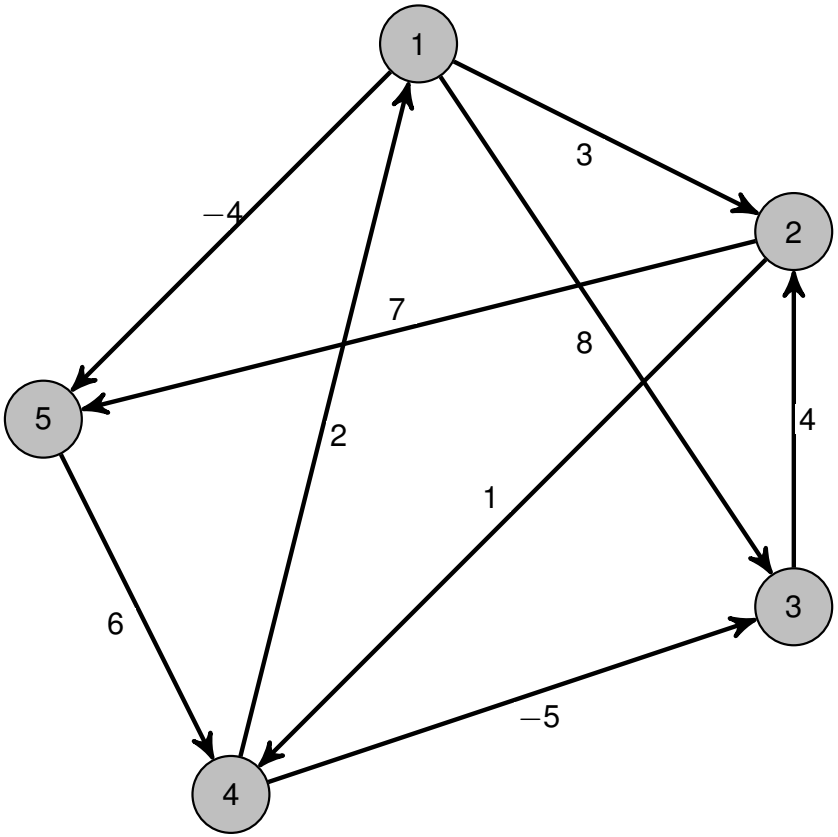
procedure ALLPAIRSHORTESTDISTANCE(W, n)
  $D^0 \leftarrow W$ *** Initialization ***
  for $k \leftarrow 1$ to n **do**
    for $i \leftarrow 1$ to n **do**
      for $j \leftarrow 1$ to n **do**
        $d_{ij}^k \leftarrow \min(d_{ij}^{k-1}, d_{ik}^{k-1} + d_{kj}^{k-1})$
      end for
    end for
  end for
  Return D^n
end procedure

Analysis

Time Complexity: $\Theta(n^3)$
Space Complexity: $\Theta(n^3)$; but reuse to get $\Theta(n^2)$ (Drop Superscript k)

Example

Graph



Weight Matrix

	1	2	3	4	5
1	0	3	8	∞	−4
2	∞	0	∞	1	7
3	∞	4	0	∞	∞
4	2	∞	−5	0	∞
5	∞	∞	∞	6	0

Example: Step 1

$D^0 = W$

	1	2	3	4	5
1	0	3	8	∞	-4
2	∞	0	∞	1	7
3	∞	4	0	∞	∞
4	2	∞	-5	0	∞
5	∞	∞	∞	6	0

D^1

	1	2	3	4	5
1	0				
2		0			
3			0		
4				0	
5					0

Example: Step 1

$$d^1_{12} = \min(d^0_{12}, d^0_{11} + d^0_{12}) = \min(3, 0 + 3) = \min(3, 3) = 3$$

$D^0 = W$

	1	2	3	4	5
1	0	3	8	∞	-4
2	∞	0	∞	1	7
3	∞	4	0	∞	∞
4	2	∞	-5	0	∞
5	∞	∞	∞	6	0

D^1

	1	2	3	4	5
1	0	3			
2		0			
3			0		
4				0	
5					0

Example: Step 1

$d^1_{12} = \min(d^0_{12}, d^0_{11} + d^0_{12}) = \min(3, 0 + 3) = \min(3, 3) = 3$
$d^1_{13} = \min(d^0_{13}, d^0_{11} + d^0_{13}) = \min(8, 0 + 8) = \min(8, 8) = 8$

$D^0 = W$

	1	2	3	4	5
1	0	3	8	∞	-4
2	∞	0	∞	1	7
3	∞	4	0	∞	∞
4	2	∞	-5	0	∞
5	∞	∞	∞	6	0

D^1

	1	2	3	4	5
1	0	3	8		
2		0			
3			0		
4				0	
5					0

Example: Step 1

$D^0 = W$

	1	2	3	4	5
1	0	3	8	∞	-4
2	∞	0	∞	1	7
3	∞	4	0	∞	∞
4	2	∞	-5	0	∞
5	∞	∞	∞	6	0

D^1

	1	2	3	4	5
1	0	3	8	∞	
2		0			
3			0		
4				0	
5					0

$$d^1_{12} = \min(d^0_{12}, d^0_{11} + d^0_{12}) = \min(3, 0 + 3) = \min(3, 3) = 3$$
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$$d^1_{14} = \min(d^0_{14}, d^0_{11} + d^0_{14}) = \min(\infty, 0 + \infty) = \min(\infty, \infty) = \infty$$

Example: Step 1

$D^0 = W$

	1	2	3	4	5
1	0	3	8	∞	-4
2	∞	0	∞	1	7
3	∞	4	0	∞	∞
4	2	∞	-5	0	∞
5	∞	∞	∞	6	0

D^1

	1	2	3	4	5
1	0	3	8	∞	-4
2		0			
3			0		
4				0	
5					0

$$d^1_{12} = \min(d^0_{12}, d^0_{11} + d^0_{12}) = \min(3, 0 + 3) = \min(3, 3) = 3$$
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$$d^1_{14} = \min(d^0_{14}, d^0_{11} + d^0_{14}) = \min(\infty, 0 + \infty) = \min(\infty, \infty) = \infty$$
$$d^1_{15} = \min(d^0_{15}, d^0_{11} + d^0_{15}) = \min(-4, 0 + -4) = \min(-4, -4) = -4$$

Example: Step 1

$D^0 = W$

	1	2	3	4	5
1	0	3	8	∞	-4
2	∞	0	∞	1	7
3	∞	4	0	∞	∞
4	2	∞	-5	0	∞
5	∞	∞	∞	6	0

D^1

	1	2	3	4	5
1	0	3	8	∞	-4
2	∞	0			
3			0		
4				0	
5					0

$$d^1_{12} = \min(d^0_{12}, d^0_{11} + d^0_{12}) = \min(3, 0 + 3) = \min(3, 3) = 3$$
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$$d^1_{21} = \min(d^0_{21}, d^0_{21} + d^0_{11}) = \min(\infty, \infty + 0) = \min(\infty, \infty) = \infty$$

Example: Step 1

$D^0 = W$

	1	2	3	4	5
1	0	3	8	∞	-4
2	∞	0	∞	1	7
3	∞	4	0	∞	∞
4	2	∞	-5	0	∞
5	∞	∞	∞	6	0

D^1

	1	2	3	4	5
1	0	3	8	∞	-4
2	∞	0	∞		
3			0		
4				0	
5					0

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Example: Step 1

$D^0 = W$

	1	2	3	4	5
1	0	3	8	∞	-4
2	∞	0	∞	1	7
3	∞	4	0	∞	∞
4	2	∞	-5	0	∞
5	∞	∞	∞	6	0

$$d^1_{12} = \min(d^0_{12}, d^0_{11} + d^0_{12}) = \min(3, 0 + 3) = \min(3, 3) = 3$$
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$$d^1_{23} = \min(d^0_{23}, d^0_{21} + d^0_{13}) = \min(\infty, \infty + 8) = \min(\infty, \infty) = \infty$$
$$d^1_{24} = \min(d^0_{24}, d^0_{21} + d^0_{14}) = \min(1, \infty + \infty) = \min(1, \infty) = 1$$

D^1

	1	2	3	4	5
1	0	3	8	∞	-4
2	∞	0	∞	1	
3			0		
4				0	
5					0

Example: Step 1

$D^0 = W$

	1	2	3	4	5
1	0	3	8	∞	-4
2	∞	0	∞	1	7
3	∞	4	0	∞	∞
4	2	∞	-5	0	∞
5	∞	∞	∞	6	0

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$$d^1_{24} = \min(d^0_{24}, d^0_{21} + d^0_{14}) = \min(1, \infty + \infty) = \min(1, \infty) = 1$$
$$d^1_{25} = \min(d^0_{25}, d^0_{21} + d^0_{15}) = \min(7, \infty + -4) = \min(7, \infty) = 7$$

D^1

	1	2	3	4	5
1	0	3	8	∞	-4
2	∞	0	∞	1	7
3			0		
4				0	
5					0

Example: Step 1

$D^0 = W$

	1	2	3	4	5
1	0	3	8	∞	-4
2	∞	0	∞	1	7
3	∞	4	0	∞	∞
4	2	∞	-5	0	∞
5	∞	∞	∞	6	0

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$$d^1_{25} = \min(d^0_{25}, d^0_{21} + d^0_{15}) = \min(7, \infty + -4) = \min(7, \infty) = 7$$
$$d^1_{31} = \min(d^0_{31}, d^0_{31} + d^0_{11}) = \min(\infty, \infty + 0) = \min(\infty, \infty) = \infty$$

D^1

	1	2	3	4	5
1	0	3	8	∞	-4
2	∞	0	∞	1	7
3	∞		0		
4				0	
5					0

Example: Step 1

$D^0 = W$

	1	2	3	4	5
1	0	3	8	∞	-4
2	∞	0	∞	1	7
3	∞	4	0	∞	∞
4	2	∞	-5	0	∞
5	∞	∞	∞	6	0

D^1

	1	2	3	4	5
1	0	3	8	∞	-4
2	∞	0	∞	1	7
3	∞	4	0		
4				0	
5					0

$$d^1_{12} = \min(d^0_{12}, d^0_{11} + d^0_{12}) = \min(3, 0 + 3) = \min(3, 3) = 3$$
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$$d^1_{14} = \min(d^0_{14}, d^0_{11} + d^0_{14}) = \min(\infty, 0 + \infty) = \min(\infty, \infty) = \infty$$
$$d^1_{15} = \min(d^0_{15}, d^0_{11} + d^0_{15}) = \min(-4, 0 + -4) = \min(-4, -4) = -4$$
$$d^1_{21} = \min(d^0_{21}, d^0_{21} + d^0_{11}) = \min(\infty, \infty + 0) = \min(\infty, \infty) = \infty$$
$$d^1_{23} = \min(d^0_{23}, d^0_{21} + d^0_{13}) = \min(\infty, \infty + 8) = \min(\infty, \infty) = \infty$$
$$d^1_{24} = \min(d^0_{24}, d^0_{21} + d^0_{14}) = \min(1, \infty + \infty) = \min(1, \infty) = 1$$
$$d^1_{25} = \min(d^0_{25}, d^0_{21} + d^0_{15}) = \min(7, \infty + -4) = \min(7, \infty) = 7$$
$$d^1_{31} = \min(d^0_{31}, d^0_{31} + d^0_{11}) = \min(\infty, \infty + 0) = \min(\infty, \infty) = \infty$$
$$d^1_{32} = \min(d^0_{32}, d^0_{31} + d^0_{12}) = \min(4, \infty + 3) = \min(4, \infty) = 4$$

Example: Step 1

$D^0 = W$

	1	2	3	4	5
1	0	3	8	∞	-4
2	∞	0	∞	1	7
3	∞	4	0	∞	∞
4	2	∞	-5	0	∞
5	∞	∞	∞	6	0

D^1

	1	2	3	4	5
1	0	3	8	∞	-4
2	∞	0	∞	1	7
3	∞	4	0	∞	
4				0	
5					0

$$d^1_{12} = \min(d^0_{12}, d^0_{11} + d^0_{12}) = \min(3, 0 + 3) = \min(3, 3) = 3$$
$$d^1_{13} = \min(d^0_{13}, d^0_{11} + d^0_{13}) = \min(8, 0 + 8) = \min(8, 8) = 8$$
$$d^1_{14} = \min(d^0_{14}, d^0_{11} + d^0_{14}) = \min(\infty, 0 + \infty) = \min(\infty, \infty) = \infty$$
$$d^1_{15} = \min(d^0_{15}, d^0_{11} + d^0_{15}) = \min(-4, 0 + -4) = \min(-4, -4) = -4$$
$$d^1_{21} = \min(d^0_{21}, d^0_{21} + d^0_{11}) = \min(\infty, \infty + 0) = \min(\infty, \infty) = \infty$$
$$d^1_{23} = \min(d^0_{23}, d^0_{21} + d^0_{13}) = \min(\infty, \infty + 8) = \min(\infty, \infty) = \infty$$
$$d^1_{24} = \min(d^0_{24}, d^0_{21} + d^0_{14}) = \min(1, \infty + \infty) = \min(1, \infty) = 1$$
$$d^1_{25} = \min(d^0_{25}, d^0_{21} + d^0_{15}) = \min(7, \infty + -4) = \min(7, \infty) = 7$$
$$d^1_{31} = \min(d^0_{31}, d^0_{31} + d^0_{11}) = \min(\infty, \infty + 0) = \min(\infty, \infty) = \infty$$
$$d^1_{32} = \min(d^0_{32}, d^0_{31} + d^0_{12}) = \min(4, \infty + 3) = \min(4, \infty) = 4$$
$$d^1_{34} = \min(d^0_{34}, d^0_{31} + d^0_{14}) = \min(\infty, \infty + \infty) = \min(\infty, \infty) = \infty$$

Example: Step 1

$D^0 = W$

	1	2	3	4	5
1	0	3	8	∞	-4
2	∞	0	∞	1	7
3	∞	4	0	∞	∞
4	2	∞	-5	0	∞
5	∞	∞	∞	6	0

D^1

	1	2	3	4	5
1	0	3	8	∞	-4
2	∞	0	∞	1	7
3	∞	4	0	∞	∞
4				0	
5					0

$$d^1_{12} = \min(d^0_{12}, d^0_{11} + d^0_{12}) = \min(3, 0 + 3) = \min(3, 3) = 3$$
$$d^1_{13} = \min(d^0_{13}, d^0_{11} + d^0_{13}) = \min(8, 0 + 8) = \min(8, 8) = 8$$
$$d^1_{14} = \min(d^0_{14}, d^0_{11} + d^0_{14}) = \min(\infty, 0 + \infty) = \min(\infty, \infty) = \infty$$
$$d^1_{15} = \min(d^0_{15}, d^0_{11} + d^0_{15}) = \min(-4, 0 + -4) = \min(-4, -4) = -4$$
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$$d^1_{24} = \min(d^0_{24}, d^0_{21} + d^0_{14}) = \min(1, \infty + \infty) = \min(1, \infty) = 1$$
$$d^1_{25} = \min(d^0_{25}, d^0_{21} + d^0_{15}) = \min(7, \infty + -4) = \min(7, \infty) = 7$$
$$d^1_{31} = \min(d^0_{31}, d^0_{31} + d^0_{11}) = \min(\infty, \infty + 0) = \min(\infty, \infty) = \infty$$
$$d^1_{32} = \min(d^0_{32}, d^0_{31} + d^0_{12}) = \min(4, \infty + 3) = \min(4, \infty) = 4$$
$$d^1_{34} = \min(d^0_{34}, d^0_{31} + d^0_{14}) = \min(\infty, \infty + \infty) = \min(\infty, \infty) = \infty$$
$$d^1_{35} = \min(d^0_{35}, d^0_{31} + d^0_{15}) = \min(\infty, \infty + -4) = \min(\infty, \infty) = \infty$$

Example: Step 1

$D^0 = W$

	1	2	3	4	5
1	0	3	8	∞	-4
2	∞	0	∞	1	7
3	∞	4	0	∞	∞
4	2	∞	-5	0	∞
5	∞	∞	∞	6	0

D^1

	1	2	3	4	5
1	0	3	8	∞	-4
2	∞	0	∞	1	7
3	∞	4	0	∞	∞
4	2			0	
5					0

$$d^1_{12} = \min(d^0_{12}, d^0_{11} + d^0_{12}) = \min(3, 0+3) = \min(3,3) = 3$$
$$d^1_{13} = \min(d^0_{13}, d^0_{11} + d^0_{13}) = \min(8, 0+8) = \min(8,8) = 8$$
$$d^1_{14} = \min(d^0_{14}, d^0_{11} + d^0_{14}) = \min(\infty, 0+\infty) = \min(\infty,\infty) = \infty$$
$$d^1_{15} = \min(d^0_{15}, d^0_{11} + d^0_{15}) = \min(-4, 0+-4) = \min(-4,-4) = -4$$
$$d^1_{21} = \min(d^0_{21}, d^0_{21} + d^0_{11}) = \min(\infty, \infty+0) = \min(\infty,\infty) = \infty$$
$$d^1_{23} = \min(d^0_{23}, d^0_{21} + d^0_{13}) = \min(\infty, \infty+8) = \min(\infty,\infty) = \infty$$
$$d^1_{24} = \min(d^0_{24}, d^0_{21} + d^0_{14}) = \min(1, \infty+\infty) = \min(1,\infty) = 1$$
$$d^1_{25} = \min(d^0_{25}, d^0_{21} + d^0_{15}) = \min(7, \infty+-4) = \min(7,\infty) = 7$$
$$d^1_{31} = \min(d^0_{31}, d^0_{31} + d^0_{11}) = \min(\infty, \infty+0) = \min(\infty,\infty) = \infty$$
$$d^1_{32} = \min(d^0_{32}, d^0_{31} + d^0_{12}) = \min(4, \infty+3) = \min(4,\infty) = 4$$
$$d^1_{34} = \min(d^0_{34}, d^0_{31} + d^0_{14}) = \min(\infty, \infty+\infty) = \min(\infty,\infty) = \infty$$
$$d^1_{35} = \min(d^0_{35}, d^0_{31} + d^0_{15}) = \min(\infty, \infty+-4) = \min(\infty,\infty) = \infty$$
$$d^1_{41} = \min(d^0_{41}, d^0_{41} + d^0_{11}) = \min(2, 2+0) = \min(2,2) = 2$$

Example: Step 1

$D^0 = W$

	1	2	3	4	5
1	0	3	8	∞	-4
2	∞	0	∞	1	7
3	∞	4	0	∞	∞
4	2	∞	-5	0	∞
5	∞	∞	∞	6	0

D^1

	1	2	3	4	5
1	0	3	8	∞	-4
2	∞	0	∞	1	7
3	∞	4	0	∞	∞
4	2	5		0	
5					0

$$d^1_{12} = \min(d^0_{12}, d^0_{11} + d^0_{12}) = \min(3, 0+3) = \min(3,3) = 3$$
$$d^1_{13} = \min(d^0_{13}, d^0_{11} + d^0_{13}) = \min(8, 0+8) = \min(8,8) = 8$$
$$d^1_{14} = \min(d^0_{14}, d^0_{11} + d^0_{14}) = \min(\infty, 0+\infty) = \min(\infty,\infty) = \infty$$
$$d^1_{15} = \min(d^0_{15}, d^0_{11} + d^0_{15}) = \min(-4, 0+-4) = \min(-4,-4) = -4$$
$$d^1_{21} = \min(d^0_{21}, d^0_{21} + d^0_{11}) = \min(\infty, \infty+0) = \min(\infty,\infty) = \infty$$
$$d^1_{23} = \min(d^0_{23}, d^0_{21} + d^0_{13}) = \min(\infty, \infty+8) = \min(\infty,\infty) = \infty$$
$$d^1_{24} = \min(d^0_{24}, d^0_{21} + d^0_{14}) = \min(1, \infty+\infty) = \min(1,\infty) = 1$$
$$d^1_{25} = \min(d^0_{25}, d^0_{21} + d^0_{15}) = \min(7, \infty+-4) = \min(7,\infty) = 7$$
$$d^1_{31} = \min(d^0_{31}, d^0_{31} + d^0_{11}) = \min(\infty, \infty+0) = \min(\infty,\infty) = \infty$$
$$d^1_{32} = \min(d^0_{32}, d^0_{31} + d^0_{12}) = \min(4, \infty+3) = \min(4,\infty) = 4$$
$$d^1_{34} = \min(d^0_{34}, d^0_{31} + d^0_{14}) = \min(\infty, \infty+\infty) = \min(\infty,\infty) = \infty$$
$$d^1_{35} = \min(d^0_{35}, d^0_{31} + d^0_{15}) = \min(\infty, \infty+-4) = \min(\infty,\infty) = \infty$$
$$d^1_{41} = \min(d^0_{41}, d^0_{41} + d^0_{11}) = \min(2, 2+0) = \min(2,2) = 2$$
$$d^1_{42} = \min(d^0_{42}, d^0_{41} + d^0_{12}) = \min(\infty, 2+3) = \min(\infty,5) = 5$$

Example: Step 1

$D^0 = W$

	1	2	3	4	5
1	0	3	8	∞	-4
2	∞	0	∞	1	7
3	∞	4	0	∞	∞
4	2	∞	-5	0	∞
5	∞	∞	∞	6	0

D^1

	1	2	3	4	5
1	0	3	8	∞	-4
2	∞	0	∞	1	7
3	∞	4	0	∞	∞
4	2	5	-5	0	
5					0

$$d^1_{12} = \min(d^0_{12}, d^0_{11} + d^0_{12}) = \min(3, 0+3) = \min(3,3) = 3$$
$$d^1_{13} = \min(d^0_{13}, d^0_{11} + d^0_{13}) = \min(8, 0+8) = \min(8,8) = 8$$
$$d^1_{14} = \min(d^0_{14}, d^0_{11} + d^0_{14}) = \min(\infty, 0+\infty) = \min(\infty,\infty) = \infty$$
$$d^1_{15} = \min(d^0_{15}, d^0_{11} + d^0_{15}) = \min(-4, 0+-4) = \min(-4,-4) = -4$$
$$d^1_{21} = \min(d^0_{21}, d^0_{21} + d^0_{11}) = \min(\infty, \infty+0) = \min(\infty,\infty) = \infty$$
$$d^1_{23} = \min(d^0_{23}, d^0_{21} + d^0_{13}) = \min(\infty, \infty+8) = \min(\infty,\infty) = \infty$$
$$d^1_{24} = \min(d^0_{24}, d^0_{21} + d^0_{14}) = \min(1, \infty+\infty) = \min(1,\infty) = 1$$
$$d^1_{25} = \min(d^0_{25}, d^0_{21} + d^0_{15}) = \min(7, \infty+-4) = \min(7,\infty) = 7$$
$$d^1_{31} = \min(d^0_{31}, d^0_{31} + d^0_{11}) = \min(\infty, \infty+0) = \min(\infty,\infty) = \infty$$
$$d^1_{32} = \min(d^0_{32}, d^0_{31} + d^0_{12}) = \min(4, \infty+3) = \min(4,\infty) = 4$$
$$d^1_{34} = \min(d^0_{34}, d^0_{31} + d^0_{14}) = \min(\infty, \infty+\infty) = \min(\infty,\infty) = \infty$$
$$d^1_{35} = \min(d^0_{35}, d^0_{31} + d^0_{15}) = \min(\infty, \infty+-4) = \min(\infty,\infty) = \infty$$
$$d^1_{41} = \min(d^0_{41}, d^0_{41} + d^0_{11}) = \min(2, 2+0) = \min(2,2) = 2$$
$$d^1_{42} = \min(d^0_{42}, d^0_{41} + d^0_{12}) = \min(\infty, 2+3) = \min(\infty,5) = 5$$
$$d^1_{43} = \min(d^0_{43}, d^0_{41} + d^0_{13}) = \min(-5, 2+8) = \min(-5,10) = -5$$

Example: Step 1

$D^0 = W$

	1	2	3	4	5
1	0	3	8	∞	-4
2	∞	0	∞	1	7
3	∞	4	0	∞	∞
4	2	∞	-5	0	∞
5	∞	∞	∞	6	0

D^1

	1	2	3	4	5
1	0	3	8	∞	-4
2	∞	0	∞	1	7
3	∞	4	0	∞	∞
4	2	5	-5	0	-2
5					0

$$d^1_{12} = \min(d^0_{12}, d^0_{11} + d^0_{12}) = \min(3, 0+3) = \min(3,3) = 3$$
$$d^1_{13} = \min(d^0_{13}, d^0_{11} + d^0_{13}) = \min(8, 0+8) = \min(8,8) = 8$$
$$d^1_{14} = \min(d^0_{14}, d^0_{11} + d^0_{14}) = \min(\infty, 0+\infty) = \min(\infty,\infty) = \infty$$
$$d^1_{15} = \min(d^0_{15}, d^0_{11} + d^0_{15}) = \min(-4, 0+-4) = \min(-4,-4) = -4$$
$$d^1_{21} = \min(d^0_{21}, d^0_{21} + d^0_{11}) = \min(\infty, \infty+0) = \min(\infty,\infty) = \infty$$
$$d^1_{23} = \min(d^0_{23}, d^0_{21} + d^0_{13}) = \min(\infty, \infty+8) = \min(\infty,\infty) = \infty$$
$$d^1_{24} = \min(d^0_{24}, d^0_{21} + d^0_{14}) = \min(1, \infty+\infty) = \min(1,\infty) = 1$$
$$d^1_{25} = \min(d^0_{25}, d^0_{21} + d^0_{15}) = \min(7, \infty+-4) = \min(7,\infty) = 7$$
$$d^1_{31} = \min(d^0_{31}, d^0_{31} + d^0_{11}) = \min(\infty, \infty+0) = \min(\infty,\infty) = \infty$$
$$d^1_{32} = \min(d^0_{32}, d^0_{31} + d^0_{12}) = \min(4, \infty+3) = \min(4,\infty) = 4$$
$$d^1_{34} = \min(d^0_{34}, d^0_{31} + d^0_{14}) = \min(\infty, \infty+\infty) = \min(\infty,\infty) = \infty$$
$$d^1_{35} = \min(d^0_{35}, d^0_{31} + d^0_{15}) = \min(\infty, \infty+-4) = \min(\infty,\infty) = \infty$$
$$d^1_{41} = \min(d^0_{41}, d^0_{41} + d^0_{11}) = \min(2, 2+0) = \min(2,2) = 2$$
$$d^1_{42} = \min(d^0_{42}, d^0_{41} + d^0_{12}) = \min(\infty, 2+3) = \min(\infty,5) = 5$$
$$d^1_{43} = \min(d^0_{43}, d^0_{41} + d^0_{13}) = \min(-5, 2+8) = \min(-5,10) = -5$$
$$d^1_{45} = \min(d^0_{45}, d^0_{41} + d^0_{15}) = \min(\infty, 2+-4) = \min(\infty,-2) = -2$$

Example: Step 1

$D^0 = W$

	1	2	3	4	5
1	0	3	8	∞	-4
2	∞	0	∞	1	7
3	∞	4	0	∞	∞
4	2	∞	-5	0	∞
5	∞	∞	∞	6	0

D^1

	1	2	3	4	5
1	0	3	8	∞	-4
2	∞	0	∞	1	7
3	∞	4	0	∞	∞
4	2	5	-5	0	-2
5	∞				0

$$d^1_{12} = \min(d^0_{12}, d^0_{11} + d^0_{12}) = \min(3, 0+3) = \min(3,3) = 3$$
$$d^1_{13} = \min(d^0_{13}, d^0_{11} + d^0_{13}) = \min(8, 0+8) = \min(8,8) = 8$$
$$d^1_{14} = \min(d^0_{14}, d^0_{11} + d^0_{14}) = \min(\infty, 0+\infty) = \min(\infty,\infty) = \infty$$
$$d^1_{15} = \min(d^0_{15}, d^0_{11} + d^0_{15}) = \min(-4, 0+-4) = \min(-4,-4) = -4$$
$$d^1_{21} = \min(d^0_{21}, d^0_{21} + d^0_{11}) = \min(\infty, \infty+0) = \min(\infty,\infty) = \infty$$
$$d^1_{23} = \min(d^0_{23}, d^0_{21} + d^0_{13}) = \min(\infty, \infty+8) = \min(\infty,\infty) = \infty$$
$$d^1_{24} = \min(d^0_{24}, d^0_{21} + d^0_{14}) = \min(1, \infty+\infty) = \min(1,\infty) = 1$$
$$d^1_{25} = \min(d^0_{25}, d^0_{21} + d^0_{15}) = \min(7, \infty+-4) = \min(7,\infty) = 7$$
$$d^1_{31} = \min(d^0_{31}, d^0_{31} + d^0_{11}) = \min(\infty, \infty+0) = \min(\infty,\infty) = \infty$$
$$d^1_{32} = \min(d^0_{32}, d^0_{31} + d^0_{12}) = \min(4, \infty+3) = \min(4,\infty) = 4$$
$$d^1_{34} = \min(d^0_{34}, d^0_{31} + d^0_{14}) = \min(\infty, \infty+\infty) = \min(\infty,\infty) = \infty$$
$$d^1_{35} = \min(d^0_{35}, d^0_{31} + d^0_{15}) = \min(\infty, \infty+-4) = \min(\infty,\infty) = \infty$$
$$d^1_{41} = \min(d^0_{41}, d^0_{41} + d^0_{11}) = \min(2, 2+0) = \min(2,2) = 2$$
$$d^1_{42} = \min(d^0_{42}, d^0_{41} + d^0_{12}) = \min(\infty, 2+3) = \min(\infty,5) = 5$$
$$d^1_{43} = \min(d^0_{43}, d^0_{41} + d^0_{13}) = \min(-5, 2+8) = \min(-5,10) = -5$$
$$d^1_{45} = \min(d^0_{45}, d^0_{41} + d^0_{15}) = \min(\infty, 2+-4) = \min(\infty,-2) = -2$$
$$d^1_{51} = \min(d^0_{51}, d^0_{51} + d^0_{11}) = \min(\infty, \infty+0) = \min(\infty,\infty) = \infty$$

Example: Step 1

$D^0 = W$

	1	2	3	4	5
1	0	3	8	∞	-4
2	∞	0	∞	1	7
3	∞	4	0	∞	∞
4	2	∞	-5	0	∞
5	∞	∞	∞	6	0

D^1

	1	2	3	4	5
1	0	3	8	∞	-4
2	∞	0	∞	1	7
3	∞	4	0	∞	∞
4	2	5	-5	0	-2
5	∞	∞			0

$$d^1_{12} = \min(d^0_{12}, d^0_{11} + d^0_{12}) = \min(3, 0+3) = \min(3,3) = 3$$
$$d^1_{13} = \min(d^0_{13}, d^0_{11} + d^0_{13}) = \min(8, 0+8) = \min(8,8) = 8$$
$$d^1_{14} = \min(d^0_{14}, d^0_{11} + d^0_{14}) = \min(\infty, 0+\infty) = \min(\infty,\infty) = \infty$$
$$d^1_{15} = \min(d^0_{15}, d^0_{11} + d^0_{15}) = \min(-4, 0+-4) = \min(-4,-4) = -4$$
$$d^1_{21} = \min(d^0_{21}, d^0_{21} + d^0_{11}) = \min(\infty, \infty+0) = \min(\infty,\infty) = \infty$$
$$d^1_{23} = \min(d^0_{23}, d^0_{21} + d^0_{13}) = \min(\infty, \infty+8) = \min(\infty,\infty) = \infty$$
$$d^1_{24} = \min(d^0_{24}, d^0_{21} + d^0_{14}) = \min(1, \infty+\infty) = \min(1,\infty) = 1$$
$$d^1_{25} = \min(d^0_{25}, d^0_{21} + d^0_{15}) = \min(7, \infty+-4) = \min(7,\infty) = 7$$
$$d^1_{31} = \min(d^0_{31}, d^0_{31} + d^0_{11}) = \min(\infty, \infty+0) = \min(\infty,\infty) = \infty$$
$$d^1_{32} = \min(d^0_{32}, d^0_{31} + d^0_{12}) = \min(4, \infty+3) = \min(4,\infty) = 4$$
$$d^1_{34} = \min(d^0_{34}, d^0_{31} + d^0_{14}) = \min(\infty, \infty+\infty) = \min(\infty,\infty) = \infty$$
$$d^1_{35} = \min(d^0_{35}, d^0_{31} + d^0_{15}) = \min(\infty, \infty+-4) = \min(\infty,\infty) = \infty$$
$$d^1_{41} = \min(d^0_{41}, d^0_{41} + d^0_{11}) = \min(2, 2+0) = \min(2,2) = 2$$
$$d^1_{42} = \min(d^0_{42}, d^0_{41} + d^0_{12}) = \min(\infty, 2+3) = \min(\infty,5) = 5$$
$$d^1_{43} = \min(d^0_{43}, d^0_{41} + d^0_{13}) = \min(-5, 2+8) = \min(-5,10) = -5$$
$$d^1_{45} = \min(d^0_{45}, d^0_{41} + d^0_{15}) = \min(\infty, 2+-4) = \min(\infty,-2) = -2$$
$$d^1_{51} = \min(d^0_{51}, d^0_{51} + d^0_{11}) = \min(\infty, \infty+0) = \min(\infty,\infty) = \infty$$
$$d^1_{52} = \min(d^0_{52}, d^0_{51} + d^0_{12}) = \min(\infty, \infty+3) = \min(\infty,\infty) = \infty$$

Example: Step 1

$D^0 = W$

	1	2	3	4	5
1	0	3	8	∞	-4
2	∞	0	∞	1	7
3	∞	4	0	∞	∞
4	2	∞	-5	0	∞
5	∞	∞	∞	6	0

D^1

	1	2	3	4	5
1	0	3	8	∞	-4
2	∞	0	∞	1	7
3	∞	4	0	∞	∞
4	2	5	-5	0	-2
5	∞	∞	∞		0

$$d^1_{12} = \min(d^0_{12}, d^0_{11} + d^0_{12}) = \min(3, 0+3) = \min(3,3) = 3$$
$$d^1_{13} = \min(d^0_{13}, d^0_{11} + d^0_{13}) = \min(8, 0+8) = \min(8,8) = 8$$
$$d^1_{14} = \min(d^0_{14}, d^0_{11} + d^0_{14}) = \min(\infty, 0+\infty) = \min(\infty,\infty) = \infty$$
$$d^1_{15} = \min(d^0_{15}, d^0_{11} + d^0_{15}) = \min(-4, 0+-4) = \min(-4,-4) = -4$$
$$d^1_{21} = \min(d^0_{21}, d^0_{21} + d^0_{11}) = \min(\infty, \infty+0) = \min(\infty,\infty) = \infty$$
$$d^1_{23} = \min(d^0_{23}, d^0_{21} + d^0_{13}) = \min(\infty, \infty+8) = \min(\infty,\infty) = \infty$$
$$d^1_{24} = \min(d^0_{24}, d^0_{21} + d^0_{14}) = \min(1, \infty+\infty) = \min(1,\infty) = 1$$
$$d^1_{25} = \min(d^0_{25}, d^0_{21} + d^0_{15}) = \min(7, \infty+-4) = \min(7,\infty) = 7$$
$$d^1_{31} = \min(d^0_{31}, d^0_{31} + d^0_{11}) = \min(\infty, \infty+0) = \min(\infty,\infty) = \infty$$
$$d^1_{32} = \min(d^0_{32}, d^0_{31} + d^0_{12}) = \min(4, \infty+3) = \min(4,\infty) = 4$$
$$d^1_{34} = \min(d^0_{34}, d^0_{31} + d^0_{14}) = \min(\infty, \infty+\infty) = \min(\infty,\infty) = \infty$$
$$d^1_{35} = \min(d^0_{35}, d^0_{31} + d^0_{15}) = \min(\infty, \infty+-4) = \min(\infty,\infty) = \infty$$
$$d^1_{41} = \min(d^0_{41}, d^0_{41} + d^0_{11}) = \min(2, 2+0) = \min(2,2) = 2$$
$$d^1_{42} = \min(d^0_{42}, d^0_{41} + d^0_{12}) = \min(\infty, 2+3) = \min(\infty,5) = 5$$
$$d^1_{43} = \min(d^0_{43}, d^0_{41} + d^0_{13}) = \min(-5, 2+8) = \min(-5,10) = -5$$
$$d^1_{45} = \min(d^0_{45}, d^0_{41} + d^0_{15}) = \min(\infty, 2+-4) = \min(\infty,-2) = -2$$
$$d^1_{51} = \min(d^0_{51}, d^0_{51} + d^0_{11}) = \min(\infty, \infty+0) = \min(\infty,\infty) = \infty$$
$$d^1_{52} = \min(d^0_{52}, d^0_{51} + d^0_{12}) = \min(\infty, \infty+3) = \min(\infty,\infty) = \infty$$
$$d^1_{53} = \min(d^0_{53}, d^0_{51} + d^0_{13}) = \min(\infty, \infty+8) = \min(\infty,\infty) = \infty$$

Example: Step 1

$D^0 = W$

	1	2	3	4	5
1	0	3	8	∞	-4
2	∞	0	∞	1	7
3	∞	4	0	∞	∞
4	2	∞	-5	0	∞
5	∞	∞	∞	6	0

D^1

	1	2	3	4	5
1	0	3	8	∞	-4
2	∞	0	∞	1	7
3	∞	4	0	∞	∞
4	2	5	-5	0	-2
5	∞	∞	∞	6	0

$$
\begin{aligned}
d^1_{12} &= \min(d^0_{12}, d^0_{11} + d^0_{12}) = \min(3, 0+3) = \min(3,3) = 3\\
d^1_{13} &= \min(d^0_{13}, d^0_{11} + d^0_{13}) = \min(8, 0+8) = \min(8,8) = 8\\
d^1_{14} &= \min(d^0_{14}, d^0_{11} + d^0_{14}) = \min(\infty, 0+\infty) = \min(\infty,\infty) = \infty\\
d^1_{15} &= \min(d^0_{15}, d^0_{11} + d^0_{15}) = \min(-4, 0+-4) = \min(-4,-4) = -4\\
d^1_{21} &= \min(d^0_{21}, d^0_{21} + d^0_{11}) = \min(\infty, \infty+0) = \min(\infty,\infty) = \infty\\
d^1_{23} &= \min(d^0_{23}, d^0_{21} + d^0_{13}) = \min(\infty, \infty+8) = \min(\infty,\infty) = \infty\\
d^1_{24} &= \min(d^0_{24}, d^0_{21} + d^0_{14}) = \min(1, \infty+\infty) = \min(1,\infty) = 1\\
d^1_{25} &= \min(d^0_{25}, d^0_{21} + d^0_{15}) = \min(7, \infty+-4) = \min(7,\infty) = 7\\
d^1_{31} &= \min(d^0_{31}, d^0_{31} + d^0_{11}) = \min(\infty, \infty+0) = \min(\infty,\infty) = \infty\\
d^1_{32} &= \min(d^0_{32}, d^0_{31} + d^0_{12}) = \min(4, \infty+3) = \min(4,\infty) = 4\\
d^1_{34} &= \min(d^0_{34}, d^0_{31} + d^0_{14}) = \min(\infty, \infty+\infty) = \min(\infty,\infty) = \infty\\
d^1_{35} &= \min(d^0_{35}, d^0_{31} + d^0_{15}) = \min(\infty, \infty+-4) = \min(\infty,\infty) = \infty\\
d^1_{41} &= \min(d^0_{41}, d^0_{41} + d^0_{11}) = \min(2, 2+0) = \min(2,2) = 2\\
d^1_{42} &= \min(d^0_{42}, d^0_{41} + d^0_{12}) = \min(\infty, 2+3) = \min(\infty,5) = 5\\
d^1_{43} &= \min(d^0_{43}, d^0_{41} + d^0_{13}) = \min(-5, 2+8) = \min(-5,10) = -5\\
d^1_{45} &= \min(d^0_{45}, d^0_{41} + d^0_{15}) = \min(\infty, 2+-4) = \min(\infty,-2) = -2\\
d^1_{51} &= \min(d^0_{51}, d^0_{51} + d^0_{11}) = \min(\infty, \infty+0) = \min(\infty,\infty) = \infty\\
d^1_{52} &= \min(d^0_{52}, d^0_{51} + d^0_{12}) = \min(\infty, \infty+3) = \min(\infty,\infty) = \infty\\
d^1_{53} &= \min(d^0_{53}, d^0_{51} + d^0_{13}) = \min(\infty, \infty+8) = \min(\infty,\infty) = \infty\\
d^1_{54} &= \min(d^0_{54}, d^0_{51} + d^0_{14}) = \min(6, \infty+\infty) = \min(6,\infty) = 6
\end{aligned}
$$

Example: Steps 2, 3, 4, 5

D^2

	1	2	3	4	5
1	0	3	8	4	-4
2	∞	0	∞	1	7
3	∞	4	0	5	11
4	2	5	-5	0	-2
5	∞	∞	∞	6	0

Example: Steps 2, 3, 4, 5

D^2

	1	2	3	4	5
1	0	3	8	4	-4
2	∞	0	∞	1	7
3	∞	4	0	5	11
4	2	5	-5	0	-2
5	∞	∞	∞	6	0

D^3

	1	2	3	4	5
1	0	3	8	4	-4
2	∞	0	∞	1	7
3	∞	4	0	5	11
4	2	-1	-5	0	-2
5	∞	∞	∞	6	0

Example: Steps 2, 3, 4, 5

D^2

	1	2	3	4	5
1	0	3	8	4	−4
2	∞	0	∞	1	7
3	∞	4	0	5	11
4	2	5	−5	0	−2
5	∞	∞	∞	6	0

D^3

	1	2	3	4	5
1	0	3	8	4	−4
2	∞	0	∞	1	7
3	∞	4	0	5	11
4	2	−1	−5	0	−2
5	∞	∞	∞	6	0

D^4

	1	2	3	4	5
1	0	3	−1	4	−4
2	3	0	−4	1	−1
3	7	4	0	5	3
4	2	−1	−5	0	−2
5	8	5	1	6	0

Example: Steps 2, 3, 4, 5

D^2

	1	2	3	4	5
1	0	3	8	4	−4
2	∞	0	∞	1	7
3	∞	4	0	5	11
4	2	5	−5	0	−2
5	∞	∞	∞	6	0

D^3

	1	2	3	4	5
1	0	3	8	4	−4
2	∞	0	∞	1	7
3	∞	4	0	5	11
4	2	−1	−5	0	−2
5	∞	∞	∞	6	0

D^4

	1	2	3	4	5
1	0	3	−1	4	−4
2	3	0	−4	1	−1
3	7	4	0	5	3
4	2	−1	−5	0	−2
5	8	5	1	6	0

D^5

	1	2	3	4	5
1	0	1	−3	2	−4
2	3	0	−4	1	−1
3	7	4	0	5	3
4	2	−1	−5	0	−2
5	8	5	1	6	0

Outline

Introduction

Problem

Given a Graph $G = (V, E)$, find its **transitive closure** defined as:

$G = (V, E^*)$ where $E^* = \{(i, j) : \text{there is path } i \rightsquigarrow j \text{ in } G\}$.

Input

Given: $G = (V, E)$ such that $|V| = n$.

Output

$n \times n$ matrix D such that $D[i][j] = d_{ij}$ such that $d_{ij} \begin{cases} 1 & \text{if there is path} i \rightsquigarrow j \\ 0 & \text{otherwise} \end{cases}$

Changes

New Weight Matrix

Assign weight of 1 or 0 to each edge. In other words, modify W as follows:

$$W[i][j] = w_{ij} \begin{cases} 1 & \text{if } (i = j) \text{ or } (i, j) \in E \\ 0 & \text{otherwise} \end{cases}$$

In effect, it is unweighted adjacency matrix.

New Operators in Recurrence Relation

Replace the operators in the recurrence relation by following boolean operators:

- $\min \rightarrow \vee$ (OR)
- $+ \rightarrow \wedge$ (AND)

Modified Recurrence Relation

For $k \geq 1$
$d_{ij}^k = d_{ij}^{k-1} \vee (d_{ik}^{k-1} \wedge d_{kj}^{k-1})$

d_{ij}^k

- $d_{ij}^k = 1$ if there is a path $i \rightsquigarrow j$ with all intermediate vertices in $\{1, 2, \cdots, k\}$, 0 otherwise.
- $d_{ij}^0 = 1$ if $i = j$ or $(i, j) \in E$, 0 otherwise.
- $d_{ij}^n = 1$ if there is a path $i \rightsquigarrow j$, 0 otherwise.

References

- Cormen, Leiserson, Rivest : *Introduction to Algorithms*