Longest Unbordered Factor in Quasilinear Time

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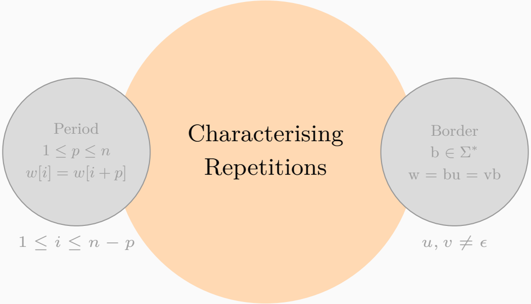
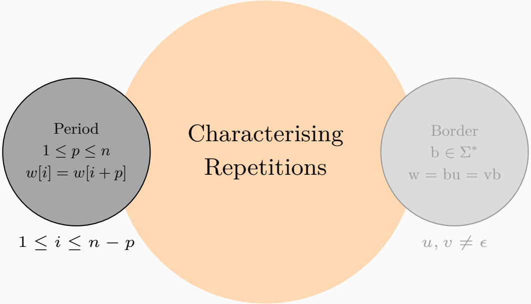
Outlines

INTRODUCTION


Characterising Repetitions
Period
$1 \leq p \leq n$
$w[i] = w[i+p]$
$1 \leq i \leq n - p$
Border
$\mathrm{b} \in \Sigma^*$
$\mathrm{w} = \mathrm{bu} = \mathrm{vb}$
$u, v \neq \epsilon$

a	a	b	a	a	b	a	a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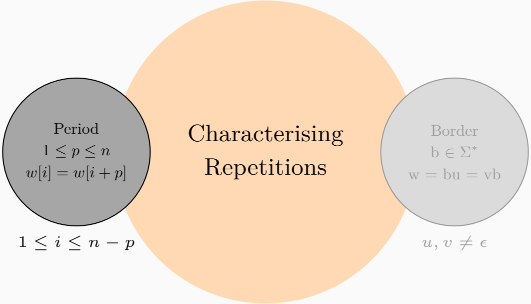
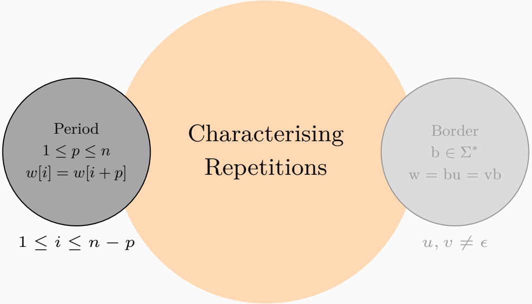
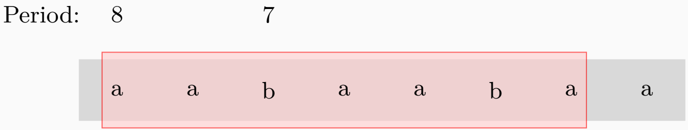


Period: 8

a a b a a b a a

Introduction





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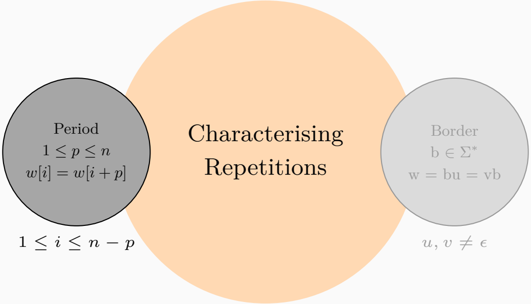

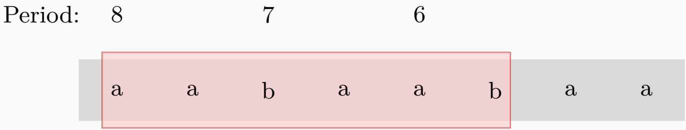

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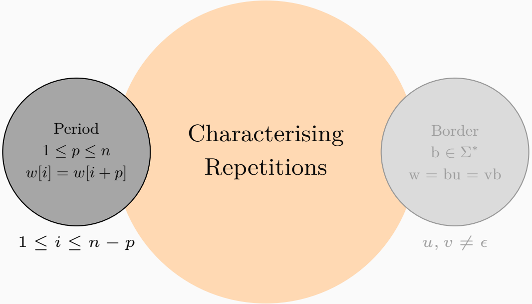

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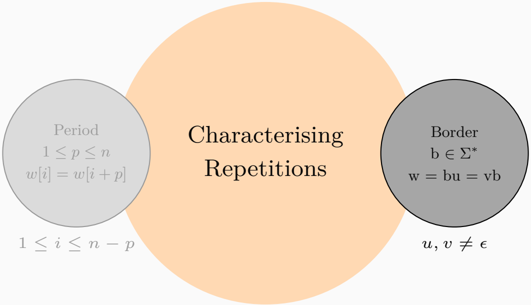
	a	a	b	a	a	b	a	a

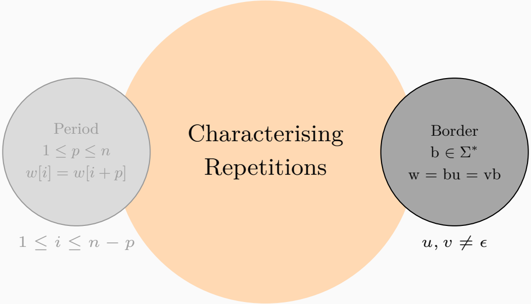


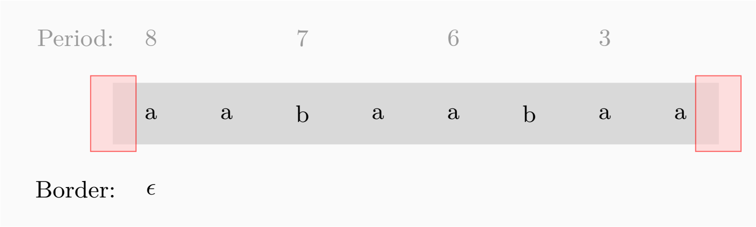
Period: 8 7 6 3

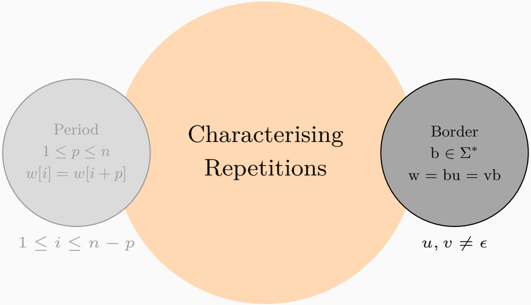
a	a	b	a	a	b	a	a

Border:

Introduction



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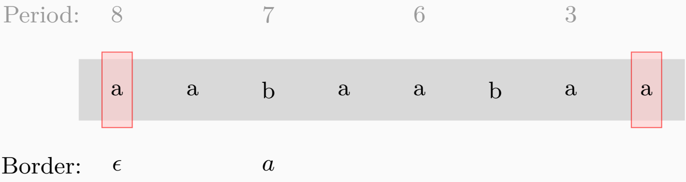
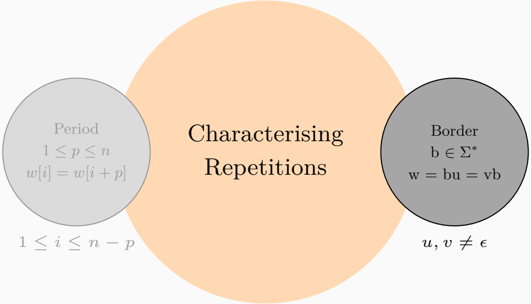

Period: 8 7 6 3
a a b a a b a a
Border: ϵ a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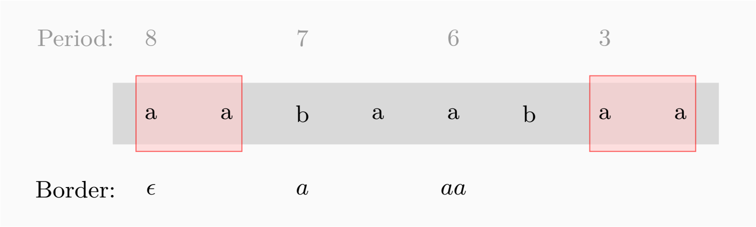
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a a b a a b a a
Border: ϵ a aa

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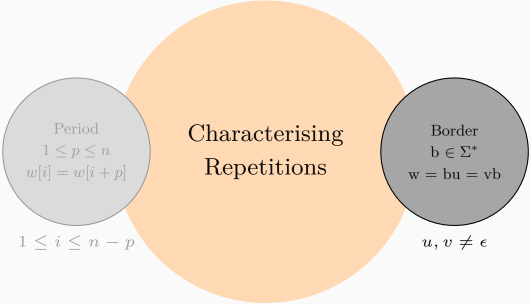

Period:	8		7		6		3	
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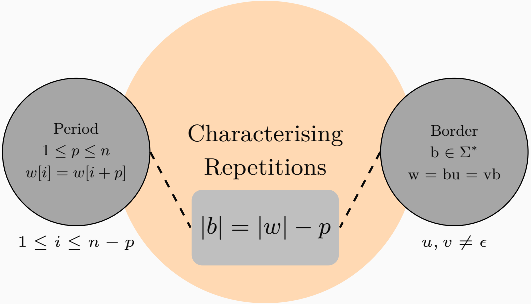
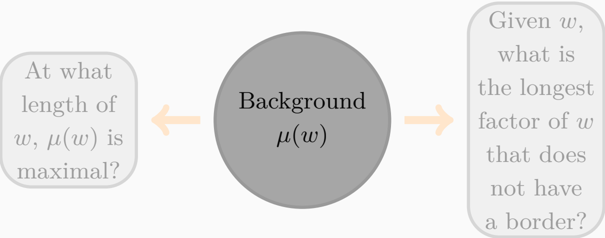


Period:	8		7		6		3	
	a	a	b	a	a	b	a	a
Border:	ϵ		a		aa		$aaabaa$	

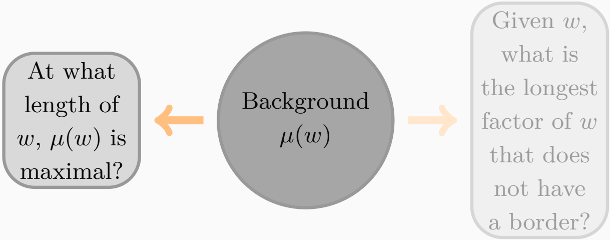
Motivation



Maximal(Longest) Unbordered Factor

- It is the longest factor of w which does not have a (non-empty) border; its length is usually represented by $\mu(w)$
- For the word $w =$ `baababa`, $\mu(w) = 5$.
- $\mu(w) \leq$ the minimal period of w.

Motivation



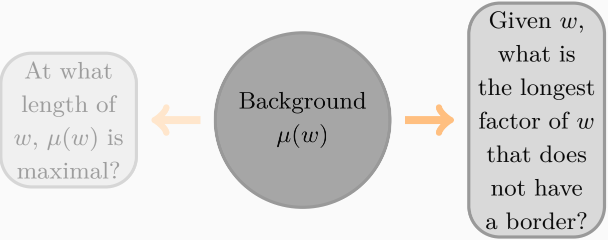
Ehrenfeucht and Silberger (1979)

⋮

Holub and Nowotka (2012)

- Asymptotically optimal upper bound ($\mu(w) \leq \frac{3}{7}n$)

Motivation



Loptev et al. (2015)

- first sub-quadratic-time (average case): $\mathcal{O}(n^2/\sigma^4)$)

Gawrychowski et al. (2015)

- Worst case $\mathcal{O}(n^{1.5})$
- $\mathcal{O}(n \log n)$ time on average
- Cording and Knudsen (2016) $\longrightarrow$ $\mathcal{O}(n)$-time [a] average-case using a refined bound on the expected length of the maximal unbordered factor

[a] improved in journal version (under review)

Our Contribution

Computing the Longest Unbordered Factor Array of a word over a general alphabet in $\mathcal{O}(n \log n)$ time with high probability.

The algorithm can also be implemented deterministically in $\mathcal{O}(n \log n \log^2 \log n)$ time.

Longest Unbordered Factor Array

Input: A word w of length n

Output: An array $\mathsf{LUF}[1..n]$ such that $\mathsf{LUF}[i]$ is the length of the maximal unbordered factor starting at position i in w, for all $1 \leq i \leq n$.

Example

i	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20
$w[i]$	a	a	b	b	a	b	a	a	b	b	a	a	b	a	b	b	a	b	a	b
LUF[i]	20	3	12	9	12	3	14	3	11	3	10	5	2	3	5	2	2	2	2	1

a a b b a b a a b b a a b a b b a b a b

1 5 15 20

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LUF[i]	20	3	12	9	12	3	14	3	11	3	10	5	2	3	5	2	2	2	2	1

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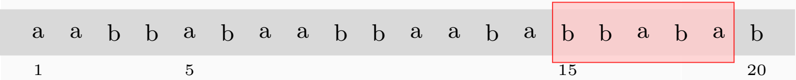

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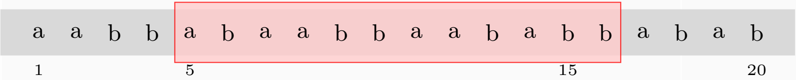

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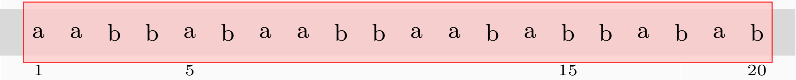

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LUF[i]	20	3	12	9	12	3	14	3	11	3	10	5	2	3	5	2	2	2	2	1

PRELIMINARIES

Duval (1982)

The shortest (non-empty) border of w is unique and unbordered.

Proposition: Duval (1982)

For any word w, there exists a unique sequence $(u_1, \cdots, u_k)$ of unbordered prefixes of w such that $w = u_k \cdots u_1$. Furthermore, the following properties hold:

(1) u_1 is the shortest border of w;

(2) u_k is the longest unbordered prefix of w;

(3) for all i, $1 \leq i \leq k$, u_i is an unbordered prefix of u_k.

unbordered-decomposition

The unique sequence described in the above proposition provides a unique unbordered-decomposition of a word.
$w = \mathsf{baababbabab} = \mathsf{baa} \cdot \mathsf{ba} \cdot \mathsf{b} \cdot \mathsf{ba} \cdot \mathsf{ba} \cdot \mathsf{b}$.

Longest Successor Factor (Length and Reference) Arrays

$$\mathsf{LSF}_\ell[i] = \begin{cases} 0 & \text{if} \quad i = n, \\ \max\{k \mid w[i\,..\,i+k-1] = w[j\,..\,j+k-1]\}, & \text{for} \quad i < j \leq n. \end{cases}$$

$$\mathsf{LSF}_r[i] = \begin{cases} nil & \text{if} \quad \mathsf{LSF}_\ell[i] = 0, \\ \max\{j \mid w[j\,..\,j+\mathsf{LSF}_\ell[i]-1] = w[i\,..\,i+\mathsf{LSF}_\ell[i]-1]\} & \text{for} \quad i < j \leq n. \end{cases}$$

Example

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$w[i]$	a	a	b	b	a	b	a	a	b	b	a	a	b	a	b	b	a	b	a	b
$\mathsf{LSF}_\ell[i]$	5	6	5	4	3	4	3	4	3	2	1	4	3	2	1	3	2	1	0	0
$\mathsf{LSF}_r[i]$	7	14	15	16	17	10	11	14	15	18	19	17	18	19	20	18	19	20	nil	nil

Longest Successor Factor (Length and Reference) Arrays

7 11 16 18 19 20

a a b b a b a a b b a a b a b b a b a b

Example

i	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20
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Computational Tools - 2

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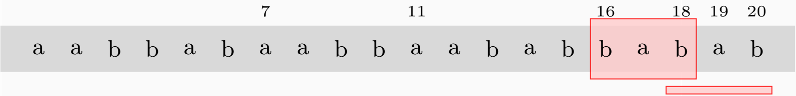

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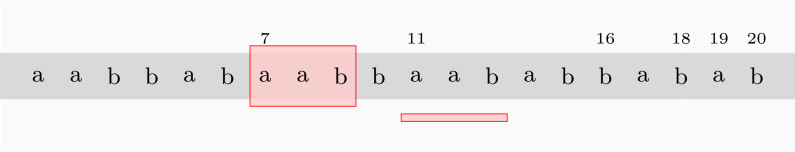

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Hook Array ($\mathsf{HOOK}[1\,..\,n]$)

At each position j, $\mathsf{HOOK}[j]$ stores the smallest position q such that the factor $w[q\,..\,j-1]$ can be decomposed into unbordered prefixes of $w[j\,..\,n]$.

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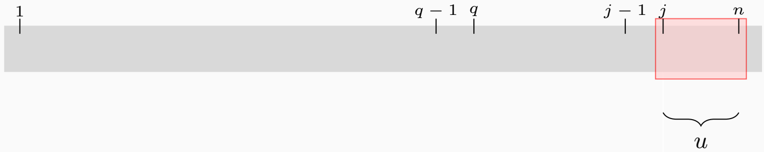

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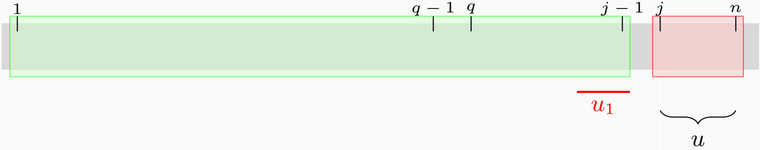

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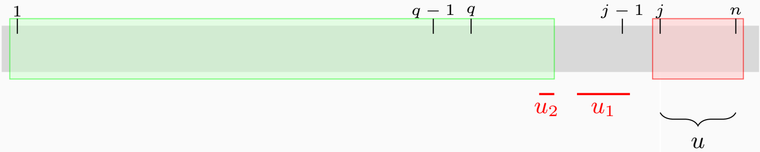

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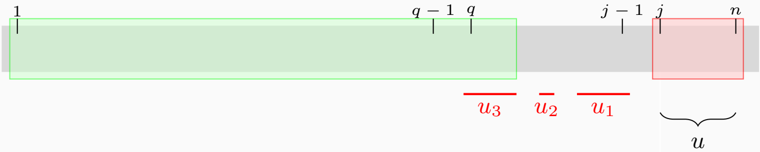

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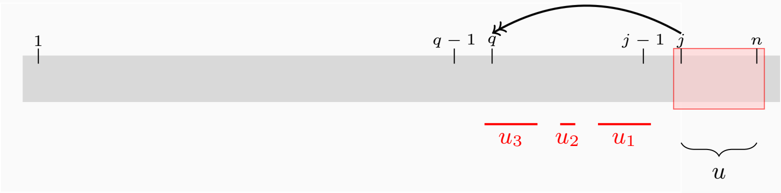

Combinatorial Tool

Hook Array (HOOK$[1\,..\,n]$)

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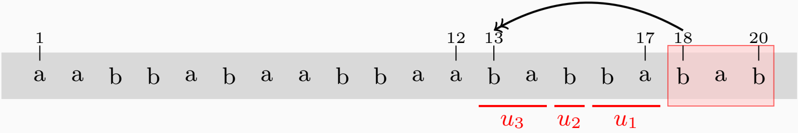

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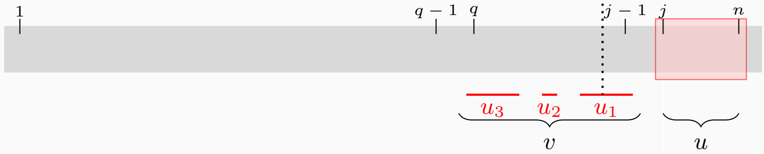
$w[i]$	a	a	b	b	a	b	a	a	b	b	a	a	b	a	b	b	a	b	a	b
HOOK$[i]$	1	1	3	3	5	3	7	1	9	3	11	11	13	1	15	13	17	13	17	20

Hook Array (HOOK$[1\,..\,n]$)

At each position j, HOOK$[j]$ stores the smallest position q such that the factor $w[q\,..\,j-1]$ can be decomposed into unbordered prefixes of $w[j\,..\,n]$.

Greedy Construction



Observation 1

The decomposition of v into unbordered prefixes of u is unique.

Observation 2

If v can be decomposed into unbordered prefixes of u, then every prefix of v also admits such a decomposition.

ALGORITHM

Case 1

If $\mathsf{LSF}_\ell[i] = 0$ then
$\mathsf{LUF}[i] = n - i + 1$, for $1 \leq i \leq n$.

Case 2

If $\mathsf{LSF}_r[i] = j$ and $\mathsf{LSF}_\ell[i] < \mathsf{LUF}[j]$ then
$\mathsf{LUF}[i] = j + \mathsf{LUF}[j] - i$, for $1 \leq i \leq n$.

High Level View

Case 1

If $\mathsf{LSF}_\ell[i] = 0$ then
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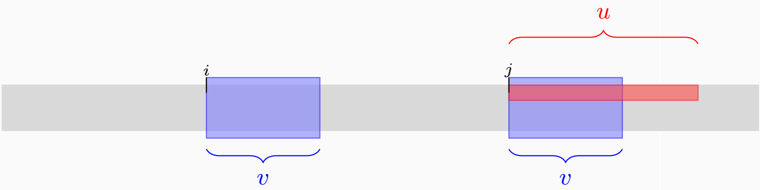

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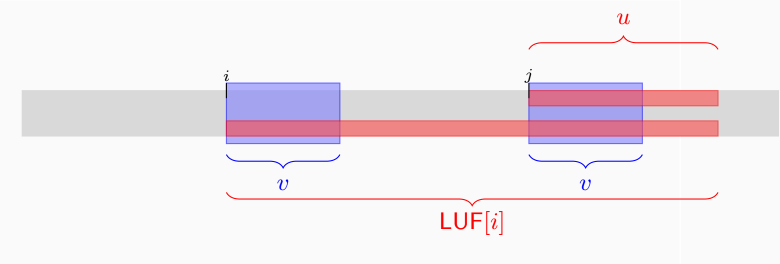

Case 3 (a)

If $\mathsf{LSF}_r[i] = j$ and $\mathsf{LSF}_\ell[i] \geq \mathsf{LUF}[j]$ then
$\mathsf{LUF}[i] = \mathsf{HOOK}[j] - i$ if $i < \mathsf{HOOK}[j]$, for $1 \leq i \leq n$.

Case 3 (b)

If $\mathsf{LSF}_r[i] = j$ and $\mathsf{LSF}_\ell[i] \geq \mathsf{LUF}[j]$ then
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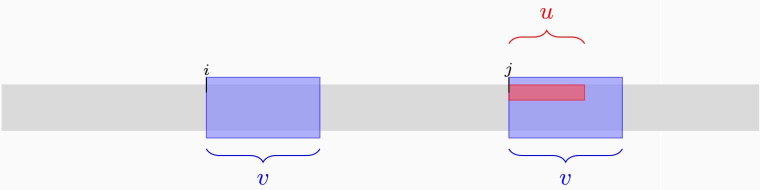

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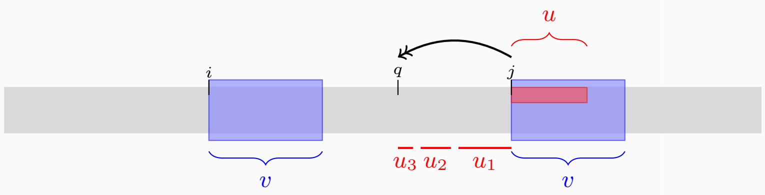

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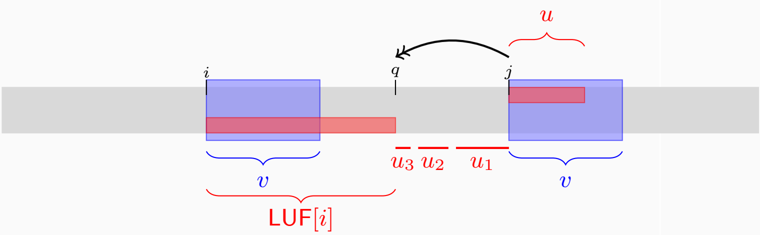

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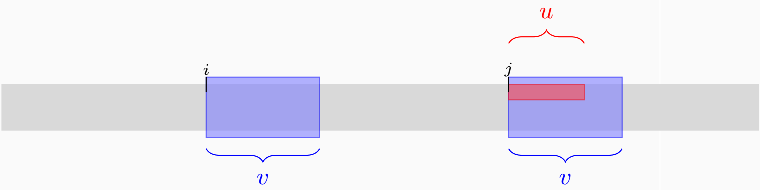

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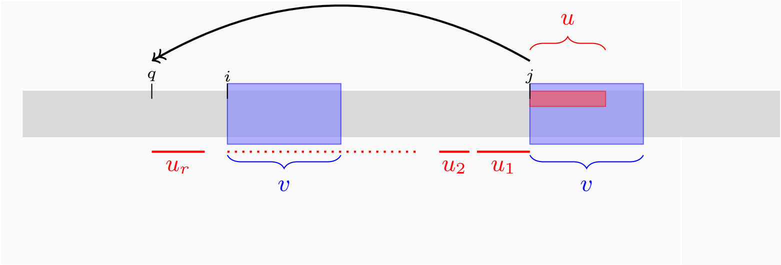

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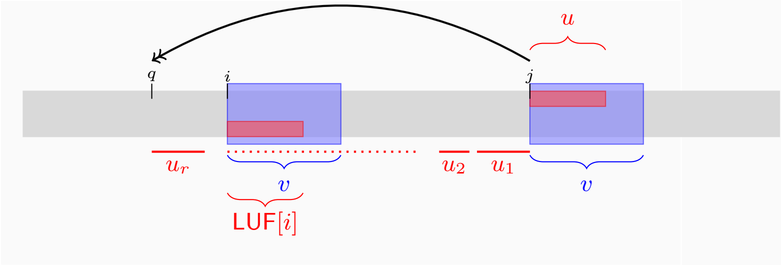
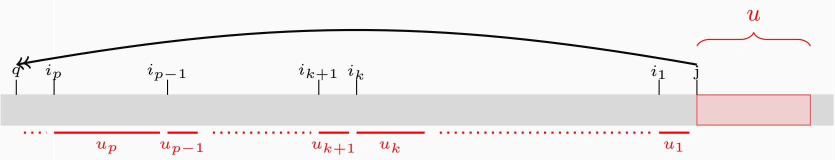


Finding Hook

Naive Construction



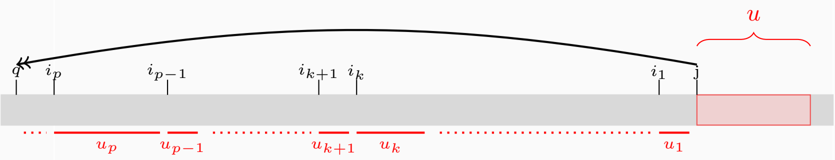
FindBeta Function

- Returns the length β of the shortest prefix of $w[j\,..\,n]$ that is a suffix of $w[1\,..\,q-1]$, or $\beta = 0$.
- Based on 'prefix-suffix queries' of Kociumaka et al. (2015, 2012): Given $d \in \mathbb{N}$; factors x & y of w, reports all prefixes of x of length between d and $2d$ that occur as suffixes of y.
- A single prefix-suffix query can be implemented in $\mathcal{O}(1)$ time after preprocessing of w which takes quasilinear[1] time.

[1] bottleneck; now solved; more later.

Finding Hook

Efficient Construction

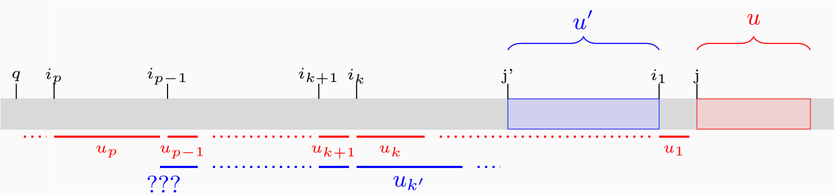


Observations

- In a chain, each u_k is unbordered. $\mathsf{LUF}[i_k] \geq |u_k| \Rightarrow \mathsf{HOOK}[i_k] \leq i_{p-1}$.
- Overlapping Chains.

Finding Hook

Efficient Construction



Observations

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- Overlapping Chains.

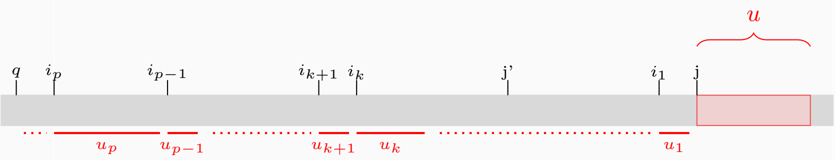
RECYCLE

Shift hook leftwards: Avoid computations between i_k and i_{p-1} w.r.t longer factors at i_k.

Genralised Hook: $\mathcal{H}_j^\ell$

$\mathcal{H}_j^0 = j$ and $\mathcal{H}_j^\ell = \mathcal{H}_j$ if $\ell \geq \mathsf{LUF}[j]$.

Efficient Construction



Implementation

- Right to left.
- Use a stack to keep track of the pairs (ℓ, i) for which the hooks $\mathcal{H}_i^\ell$ need to be determined.
- Update values in HOOK.

ANALYSIS

Twin Set

Purpose

- Correctness
- Running time analysis
- Efficient FindBeta

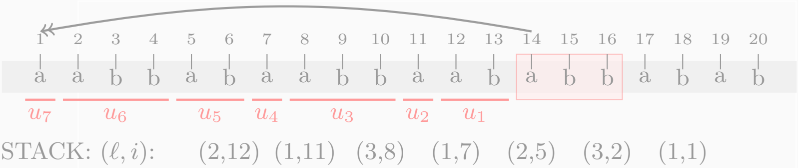
Definition: $\mathcal{T}_j^\ell$

$$\mathcal{T}_j^\ell = \{i \mid (\ell, i) \text{ was pushed onto the stack of } j\}.$$

Characteristics

- $\mathcal{S}_j = \bigcup_{\ell=1}^{\mathsf{LUF}[j]} \mathcal{T}_j^\ell$.
- A unique shortest unbordered prefix of $w[j\mathinner{.\,.}\mathsf{LUF}[j]-1]$ occurs at each i belonging to the same twin set.
- Dynamic: Parent, Base

Twin Set



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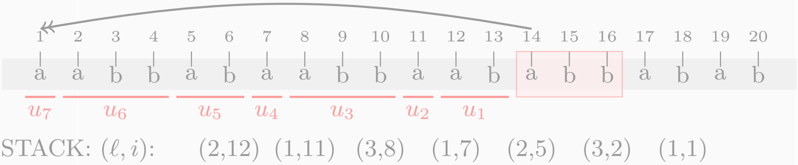

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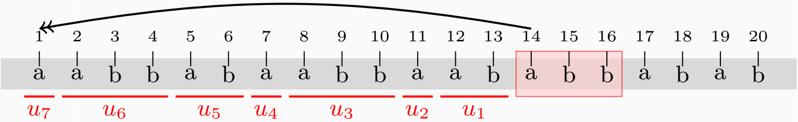

STACK: (ℓ, i): (2,12) (1,11) (3,8) (1,7) (2,5) (3,2) (1,1)

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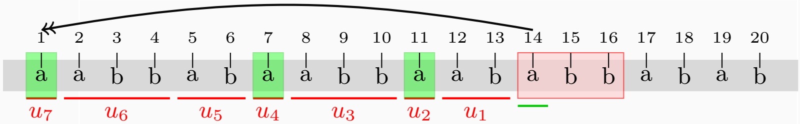

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$\mathcal{T}_{14}^{1} = \{11, 7, 1\}$

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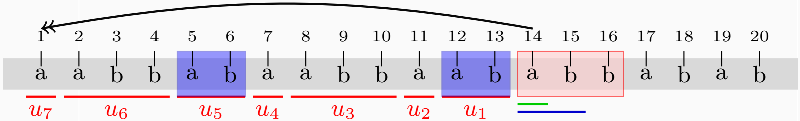

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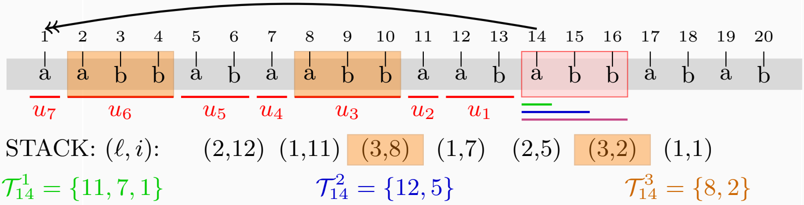

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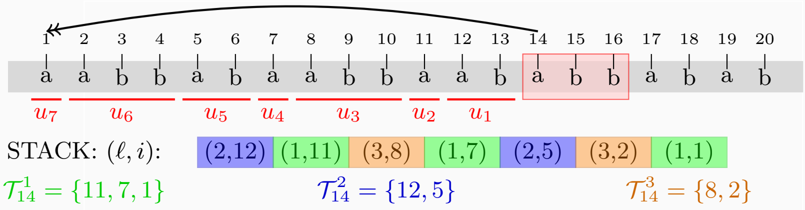

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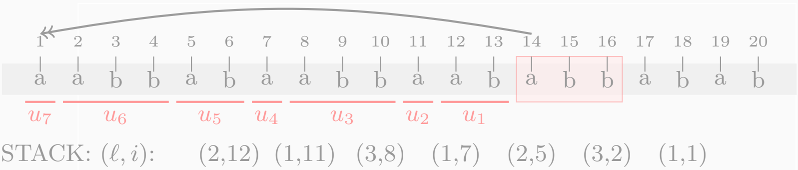

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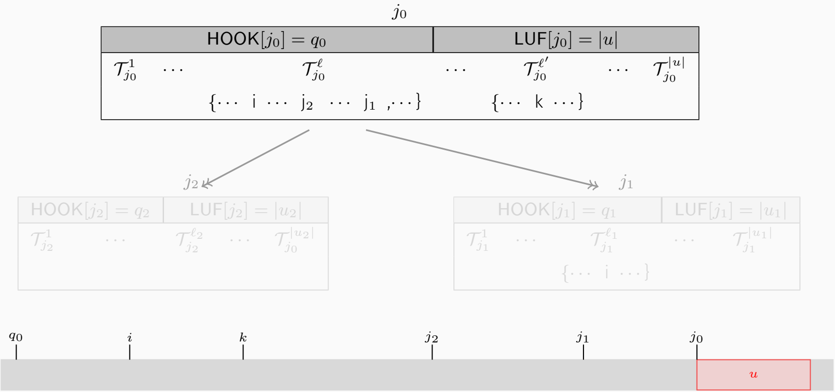
Behaviour

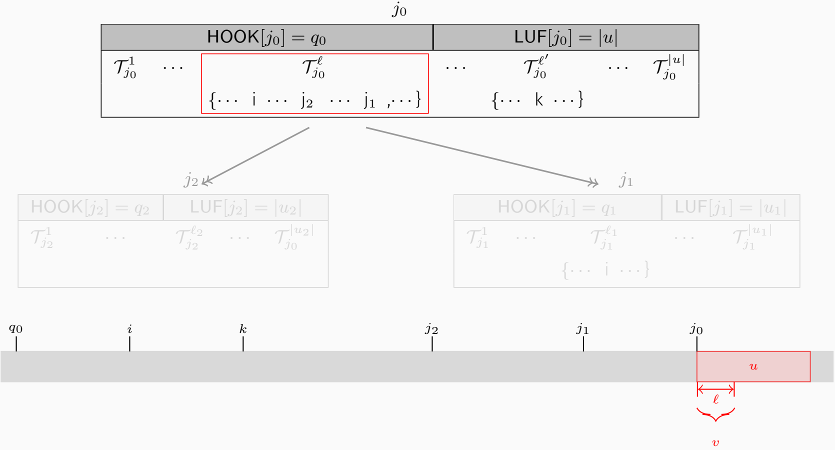


If j_0 and j_1 are two references such that j_0 is the parent of j_1 and $j_1 \in \mathcal{T}^\ell_{j_0}$, then each position $i \in \mathcal{S}_{j_1}$ satisfies the following properties:

(1) $i \in \mathcal{T}^\ell_{j_0}$;

(2) there exists $k \in \mathcal{T}^{\ell'}_{j_0}$, with $\ell' > \ell$, such that $(k + \ell' - i, i)$ is pushed onto the stack of j_1.

Behaviour



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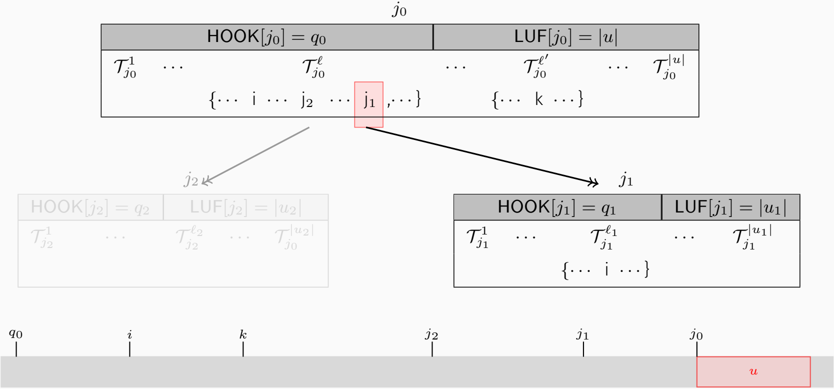

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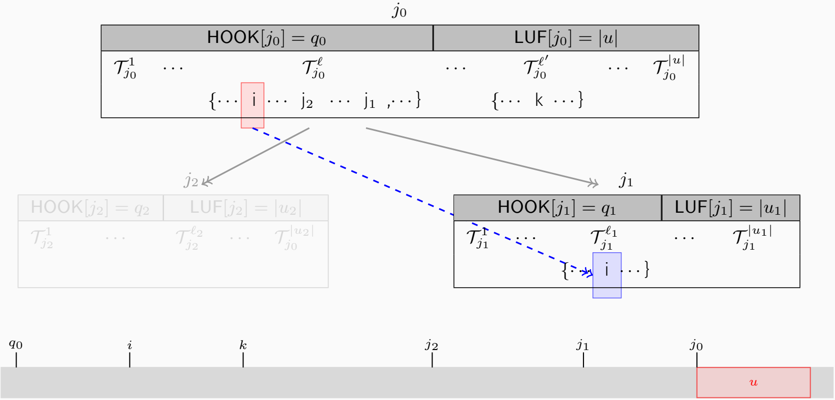

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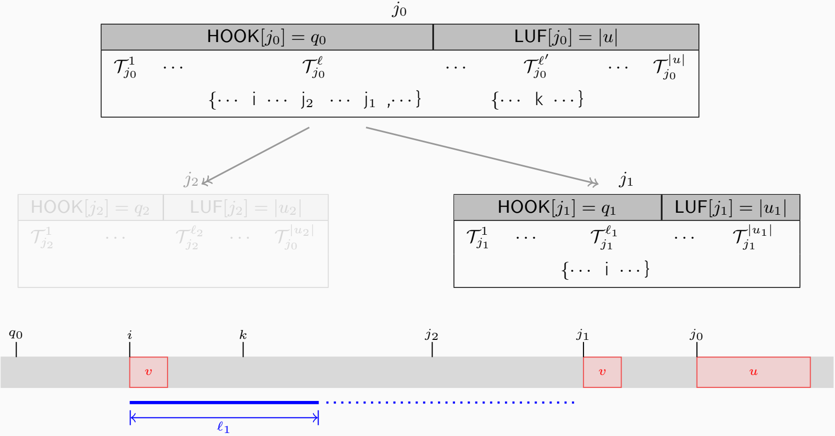

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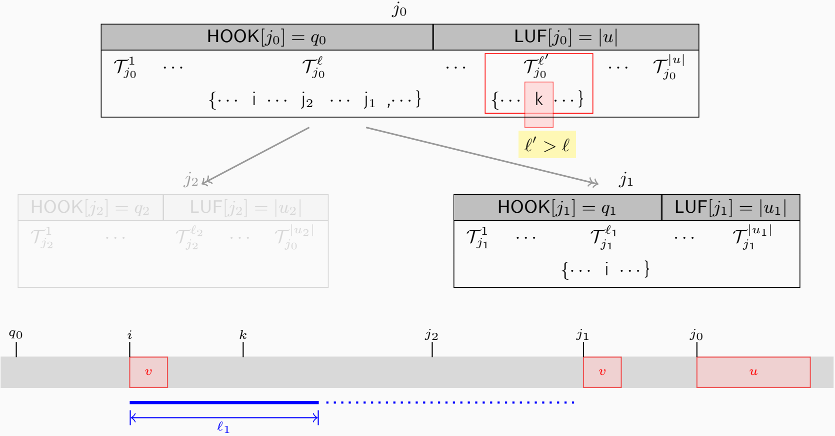

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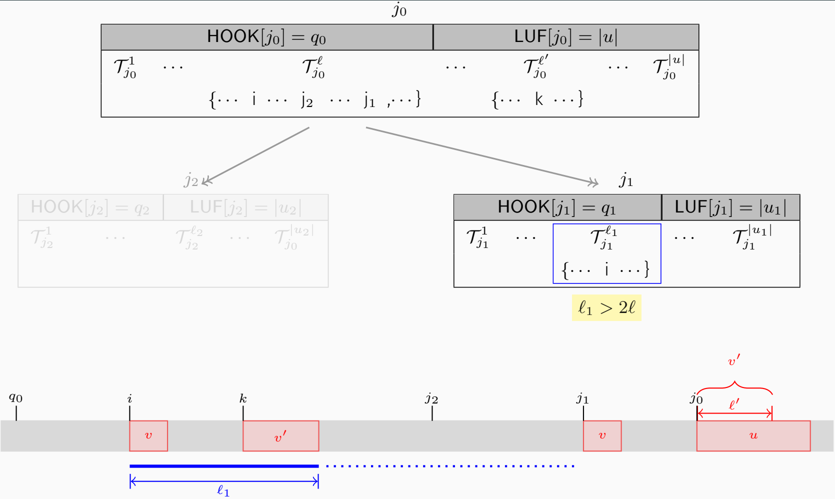

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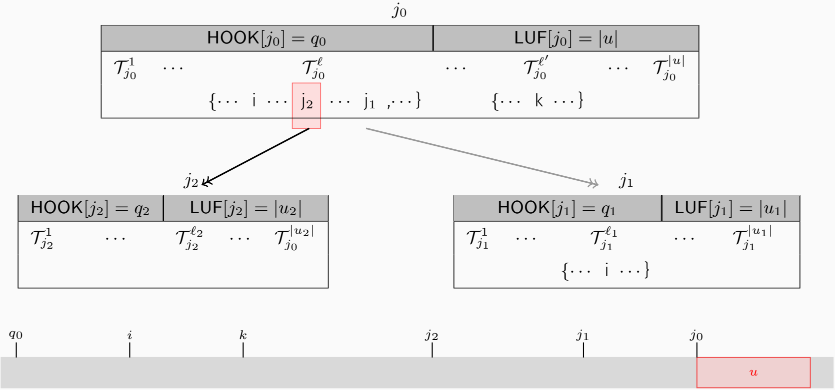
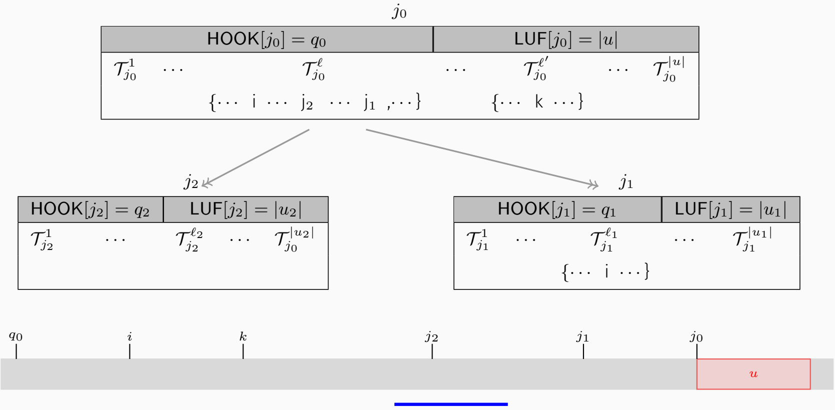


If j_0 is the parent of two references $j_2 < j_1$, both of which belong to $\mathcal{T}^\ell_{j_0}$, then $\mathcal{S}_{j_1} \cap \mathcal{S}_{j_2} = \emptyset$.

Behaviour



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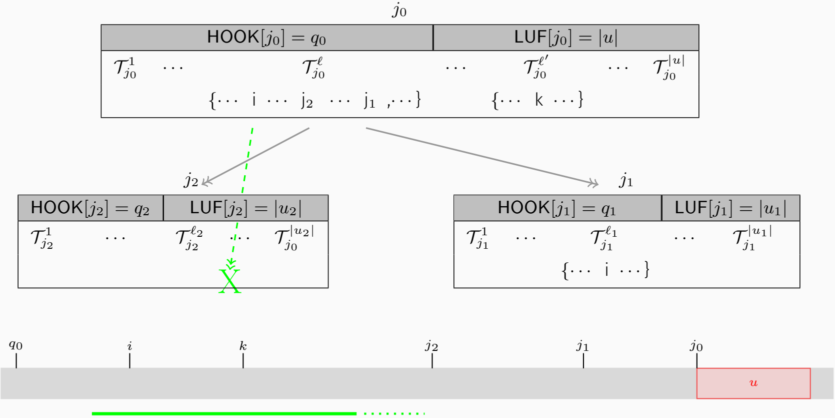

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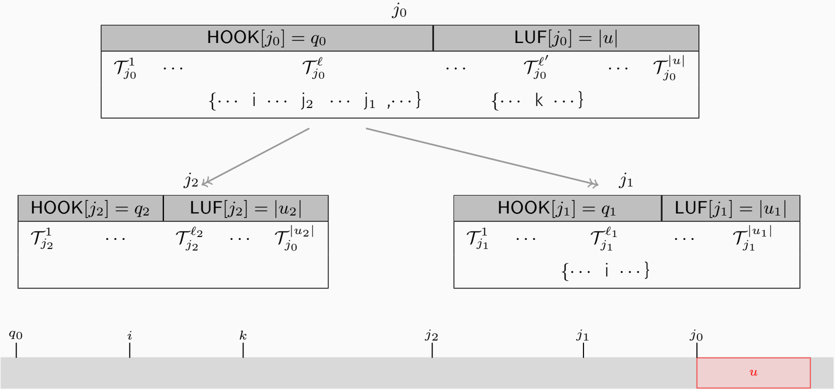


If $j_2 < j_1$ are two base references then $\mathcal{S}_{j_1} \cap \mathcal{S}_{j_2} = \emptyset$.

Analysis

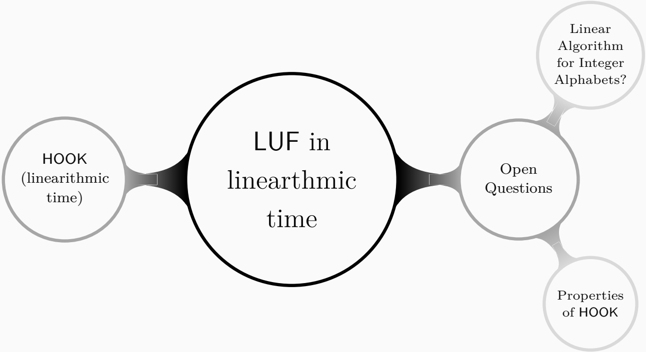
- The total size of all the stacks used throughout the algorithm is $\mathcal{O}(n \log n)$.
- The total running time of the FindBeta function is $\mathcal{O}(n \log n)$
  - Start from $d = 2\ell$ for prefix-suffix queries if the reference's parent twin-set is of length $= \ell$.

Given a word w of length n, our algorithm solves the Longest Unbordered Factor Array problem in $\mathcal{O}(n \log n)$ time with high probability. It can also be implemented deterministically in $\mathcal{O}(n \log n \log^2 \log n)$ time. [1]

[1] Update: Deterministically in $\mathcal{O}(n \log n)$ after the proposed linear time construction of the data structure to answer constant-time prefix-suffix query in Kociumaka (2018).

SUMMARY

Summary


HOOK (linearithmic time)
LUF in linearthmic time
Open Questions
Linear Algorithm for Integer Alphabets?
Properties of HOOK

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Thank You!

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